University of Nottingham
Department of Mechanical Engineering

MM2MS2 Mechanics of Solids 2
Exercise Sheet 6 – Fatigue & Fracture Solutions

1. A circular connecting rod is made from steel having the following properties:

Ultimate Tensile Stress, σ_u:	950MPa
Yield Stress, σ_y:	800MPa
Fatigue Endurance Limit, σ_e:	500MPa

The rod is subjected to a fully reversed axial load of 180kN. Determine the minimum rod diameter, allowing a Factor of Safety of 2, if the rod end produces a Fatigue Strength Reduction Factor of 2.1, where the Stress Concentration Factor is 2.5.

[Ans: 43.88mm]

Solution 1

Let Rod Diameter = D.

Nominal Alternating Stress in the rod is given as:

$$\sigma_{nom} = \frac{\sigma_{max}}{A} = \frac{180{,}000 \times 4}{\pi D^2} \tag{1}$$

Where Nominal Stress, σ_{nom}, must have a maximum value of the Fatigue Endurance Limit in order to avoid failure.

For a plain specimen:

$$\sigma_e = 500MPa$$

For a notched specimen, with Fatigue Strength Reduction Factor, k_f, applied:

$$\sigma_e = \frac{500}{k_f} = \frac{500}{2.1} MPa$$

For a notched specimen, with Fatigue Strength Reduction Factor, k_f, and Safety Factor, F, applied:

$$\sigma_e = \frac{500}{k_f \times F} = \frac{500}{2.1 \times 2} = 119.05MPa \tag{2}$$

Substituting (2) into (1), bearing in mind that the Nominal Stress must be equal to or less than this value, therefore gives:

$$119.05 \geq \frac{180{,}000 \times 4}{\pi D^2}$$

Rearranging:

$$\therefore D^2 \geq \frac{180{,}000 \times 4}{\pi \times 119.05} = 1925.1mm^2$$

$$\therefore D \geq \sqrt{1925.1} = 43.88mm$$

This is the minimum value of rod diameter to guard against fatigue failure.

Checking yielding (connecting rod is no use if plastically deformed). Equation (1) gives the Nominal Stress which must be less than the Yield Stress with the Safety Factor and Stress Concentration Factor. I.e.:

$$\frac{800}{2 \times 2.5} \geq \frac{180{,}000 \times 4}{\pi D^2}$$

$$\therefore D^2 \geq \frac{180{,}000 \times 4 \times 2 \times 2.5}{\pi \times 800} = 1432.39$$

$$\therefore D \geq \sqrt{1432.39} = 37.84mm$$

This is the minimum value of rod diameter to guard against yielding.

Therefore the choice must be for the minimum rod diameter to be 43.88mm in order to prevent both yielding and fatigue failure.

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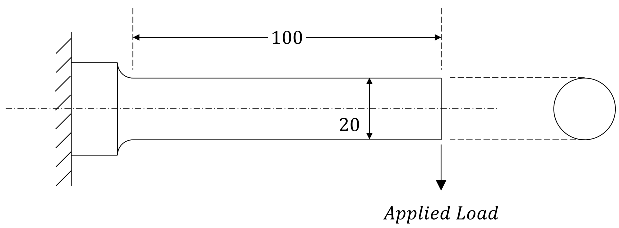
2. A Mild Steel cantilever beam of circular cross-section is subjected to a load at its free end (as shown in Fig Q2 which varies cyclically from P to $-3P$. Determine the maximum value of P if the Fatigue Strength Reduction Factor for the fillet is 1.85 and a Safety Factor of 2 is assumed. (Hint: use the Goodman diagram and apply the Fatigue Strength Reduction Factor to the stress amplitude only).



All dimensions in mm

Fig Q2

[Ans: 188.57N]

Solution 2

We shall design to the line from:

$$\frac{\sigma_u}{F}$$

on the Mean Stress, σ_m, axis of the Goodman diagram to:

$$\frac{\sigma_e}{F \times k_f}$$

on the Alternating Stress, σ_a, axis.

I.e. the Ultimate Tensile Stress, σ_u, is reduced by the required Safety Factor, F (=2) and the Fatigue Endurance Limit, σ_e, is reduced by a by the Safety Factor and the Fillet Reduction Factor, k_f (=1.85), as shown in Fig Q2.1.

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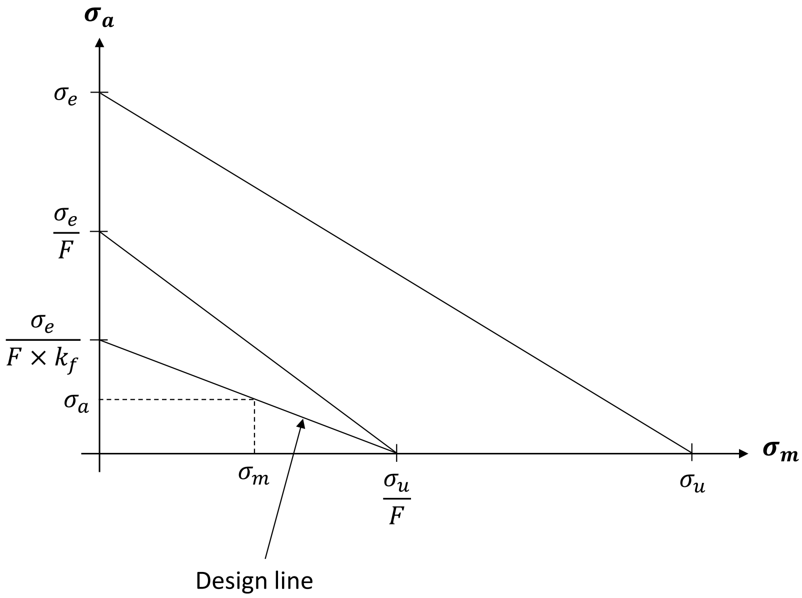

Fig Q2.1 Goodman diagram indicating the effects of the Safety Factor, F, and the Fillet Reduction Factor, k_f.

From similar triangles:

$$\frac{\sigma_a}{\left(\frac{\sigma_u}{F} - \sigma_m\right)} = \frac{\left(\frac{\sigma_e}{F \times k_f}\right)}{\left(\frac{\sigma_u}{F}\right)}$$

$$\therefore \sigma_a = \frac{\sigma_e}{F \times k_f} - \frac{\sigma_e \sigma_m}{k_f \times \sigma_u} \quad (1)$$

In this case the stress alternates such that:

$$\sigma_a = 2\sigma_m \quad (2)$$

as shown in Fig Q2.2.

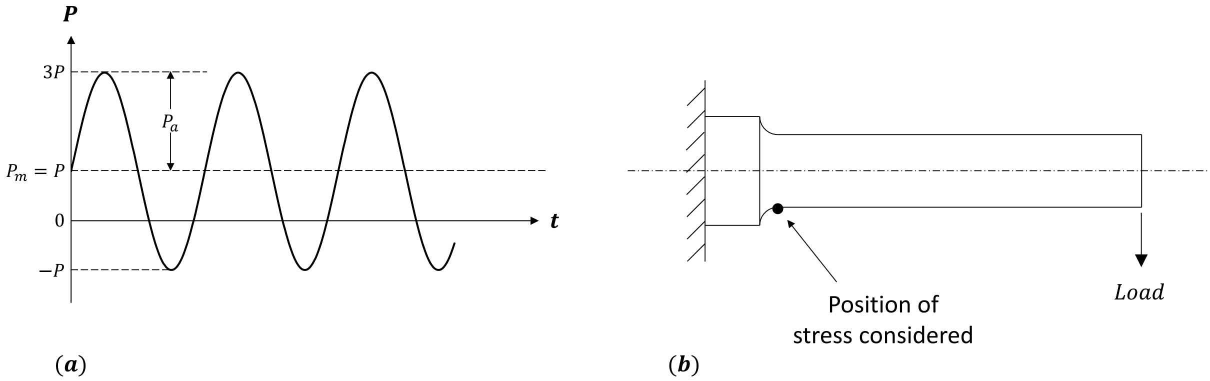

Fig Q2.2 (a) Loading profile for the shaft and (b) position of the stress considered.

Fig Q2.2 is drawn with the loads varying from $-P$ to $3P$ (as opposed to P to $-3P$) as tensile stresses are more harmful than compressive stress, and both compressive and tensile stresses are present in bending.

Assuming material values for Mild Steel as:

- Ultimate Tensile Stress, $\sigma_u = 430MPa$
- Fatigue Endurance Limit, $\sigma_e = 200MPa$

Substituting equation (2) into equation (1) and applying values of F, K_f, S_e and S_u gives:

$$2S_m = \frac{S_e}{2 \times 1.85} - \frac{S_e S_m}{1.85 \times S_u} = \frac{200}{2 \times 1.85} - \frac{200 S_m}{1.85 \times 430} = 54.05 - 0.251 S_m$$

$$\therefore S_m(2 + 0.251) = 54.05$$

$$\therefore S_m = \frac{54.05}{(2 + 0.251)} = 24.01MPa$$

From the Beam Bending Equation:

$$\frac{M}{I} = \frac{\sigma}{y} \quad (3)$$

Where, in this case:

$$M = P \times L = 100P \quad (4)$$

and for a circular section,

$$I = \frac{\pi D^4}{64} \tag{5}$$

Substituting (4) and (5) into (3) and setting σ to σ_m gives:

$$\frac{100P}{\frac{\pi D^4}{64}} = \frac{S_m}{0.01}$$

$$\therefore S_m(= 24.01) = \frac{100P \times 0.01}{\frac{\pi \times 0.02^4}{64}}$$

$$\therefore P = \frac{24.01 \times \frac{\pi \times 20^4}{64}}{100 \times 10} = 188.57N$$

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3. A circular steel shaft having a transverse oil hole is subjected to a torsional load which varies from $-100Nm$ to $400Nm$ (i.e. values in opposite senses). Determine the necessary shaft diameter assuming that the hole causes a Fatigue Strength Reduction Factor of 1.75 and making use of a Factor of Safety of 1.5. Assume the following properties for steel:

Ultimate Shear Stress, τ_u	400MPa
Fatigue Endurance Limit, τ_e	260MPa

[Ans: 25.02mm]

Solution 3

We shall design to the line from:

$$\frac{\tau_u}{F}$$

on the Mean Shear Stress, τ_m, axis of the Goodman diagram to:

$$\frac{\tau_e}{F \times k_f}$$

on the Alternating Shear Stress, τ_a, axis.

I.e. the Ultimate Shear Stress, τ_u, is reduced by the required Safety Factor, F (=1.5), and the Fatigue Endurance Limit, τ_e, is reduced by a by the Safety Factor and the Fatigue Strength Reduction Factor, k_f (=1.75), as shown in Fig Q3.1.

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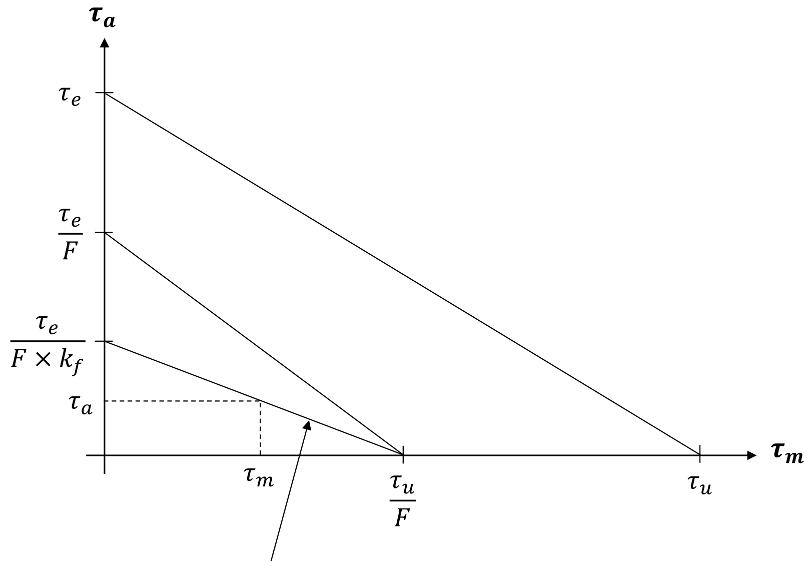

Fig Q3.1 Goodman diagram indicating the effects of the Safety Factor, F, and the Fatigue Strength Reduction Factor, k_f.

From similar triangles:

$$\frac{\tau_a}{\left(\frac{\tau_u}{F} - \tau_m\right)} = \frac{\left(\frac{\tau_e}{F \times k_f}\right)}{\left(\frac{\tau_u}{F}\right)}$$

$$\therefore \tau_a = \frac{\left(\frac{\tau_e}{F \times k_f}\right)}{\left(\frac{\tau_u}{F}\right)} \times \left(\frac{\tau_u}{F} - \tau_m\right) = \frac{\frac{\tau_e \tau_u}{F^2 k_f} - \frac{\tau_e \tau_m}{F k_f}}{\frac{\tau_u}{F}} = \frac{\tau_e}{F k_f} - \frac{\tau_e \tau_m}{k_f \tau_u}$$

$$\therefore \tau_a = \frac{260}{1.5 \times 1.75} - \frac{260\tau_m}{1.75 \times 400} = 99.05 - 0.371\tau_m \quad (1)$$

In this case the stress alternates such that:

$$T_a = 1.67T_m$$

as shown in Fig Q3.2.

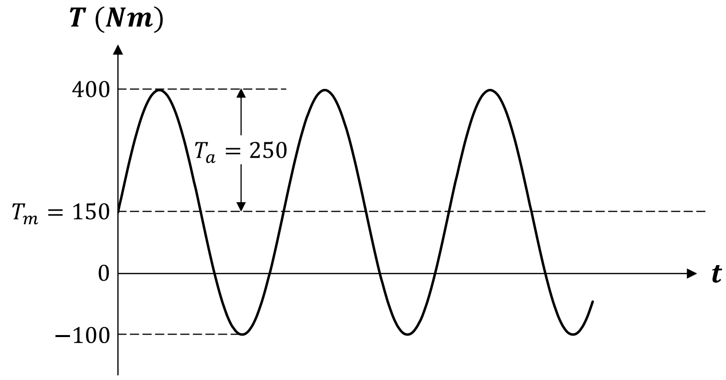

Fig Q3.2 Loading profile for the shaft.

Where, due to the proportionality of T and T as demonstrated by the Torsion equation (rotational equivalent to the Beam Bending Equation) for constant values of J and r:

$$\tau_a = 1.67\tau_m \quad (2)$$

Torsion equation:

$$\frac{T}{J} = \frac{\tau}{r}$$

$$\therefore \tau = \frac{Tr}{J} \quad (3)$$

Substituting equation (2) into equation (1) gives:

$$1.67\tau_m = 99.05 - 0.371\tau_m$$

$$\therefore \tau_m(1.67 + 0.371) = 99.05$$

$$\therefore \tau_m = \frac{99.05}{(1.67 + 0.371)} = 48.53MPa \quad (4)$$

From equation (3):

$$\therefore \tau = \frac{Tr}{J} = \frac{T \times \frac{D}{2}}{\left(\frac{\pi D^4}{32}\right)} = \frac{16T}{\pi D^3} \tag{5}$$

I.e. the Nominal Shear Stress, where:

$$r = \frac{D}{2}$$

and,

$$J = \frac{\pi D^4}{32}$$

Substituting (4) into (5) and setting τ to τ_m gives:

$$\tau_m (= 48.53MPa) = \frac{16T}{\pi D^3}$$

$$\therefore D^3 = \frac{16 \times 150{,}000}{\pi \times 48.53} = 15{,}741.68mm^3$$

$$\boxed{\therefore D = 25.06mm}$$

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4.

(a) Determine the maximum size of a surface crack or flaw that may exist in the Aluminium panel shown in Fig Q4 without affecting unstable crack propagation. The applied tensile load, σ_{nom}, is equal to half of the material yield stress, σ_y. Assume linear elastic material and that Fracture Toughness is given as:

$$K_I = \sigma_{nom}\sqrt{\pi a}$$

Assume:

$$\sigma_y = 280MPa$$

$$K_{I_c} = 32MPa\sqrt{m}$$

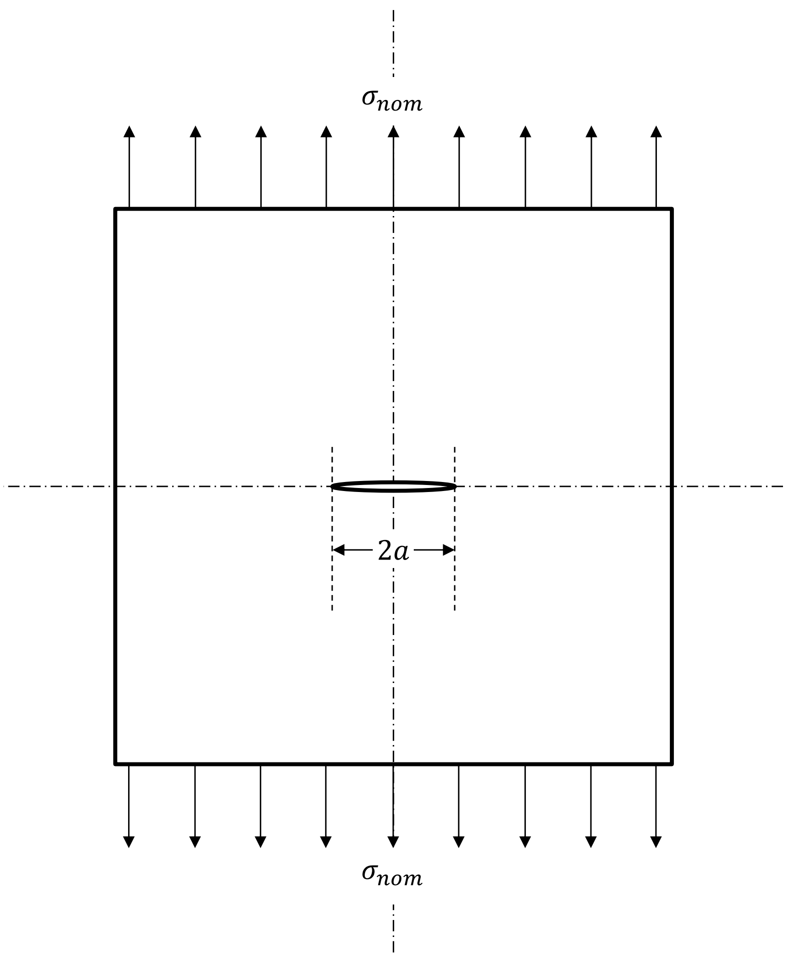

Fig Q4

(b) It is found during inspection that there is a crack contained in the panel, size $a = 1mm$. If the external load is fluctuated such that the maximum nominal stress is half of the yield point and the minimum stress is compressive of 30 MPa, estimate the life cycle of the structure.

[Ans: a) 16.6mm, b) 4,623,661 cycles]

Solution 4

(a)

$$\sigma_{nom} = \frac{1}{2}\sigma_y = 140MPa$$

It is given that:

$$K_I = \sigma_{nom}\sqrt{\pi a}$$

Therefore,

$$K_{I_c} = \sigma_{nom}\sqrt{\pi a_c}$$

(c = critical)

Where $K_{I_c} = 32MPa$.

$$\therefore 32 = 140\sqrt{\pi a_c}$$

$$\therefore a_c = \left(\frac{32}{140}\right)^2 \times \frac{1}{\pi} = 0.0166m = 16.6mm$$

(b) The applied load waveform is as shown in Fig Q4.1 (note that $\sigma_{min} = 0MPa$, not $-30MPa$, as compressive stress does not contribute to crack propagation):

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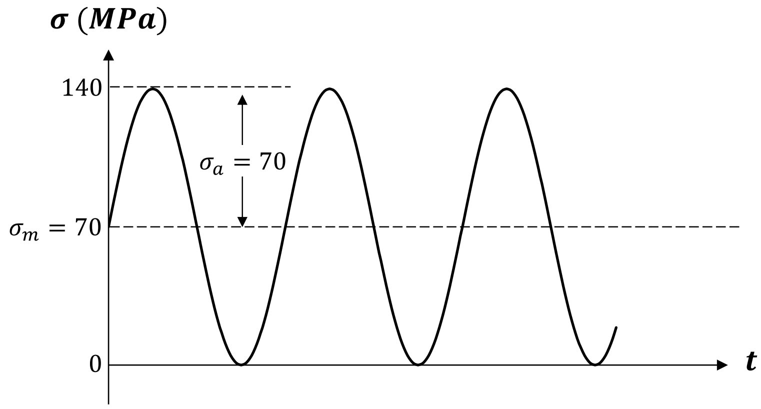

Fig Q4.1 Loading waveform

The Paris Law is given as:

$$\frac{da}{dN} = C\Delta K^m \qquad (1)$$

Where:

$$\Delta K = \Delta\sigma\sqrt{\pi a} \qquad (2)$$

as given in the question.

Substituting equation (2) into equation (1) gives:

$$da = C\left(\Delta\sigma\sqrt{\pi a}\right)^m dN$$

$$\therefore dN = \frac{1}{C\left(\Delta\sigma\sqrt{\pi a}\right)^m} da$$

$$\therefore \int_0^N dN = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m} \int_{a_i}^{a_c} \frac{1}{\left(\sqrt{a}\right)^m} da \left(= \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m} \int_{a_i}^{a_c} \frac{1}{a^{\frac{m}{2}}} da = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m} \int_{a_i}^{a_c} a^{-\frac{m}{2}} da \right)$$

(i = initial)

$$\therefore [N]_0^N = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m} \left[\frac{a^{1-\frac{m}{2}}}{1-\frac{m}{2}}\right]_{a_i}^{a_c} \left(= \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m} \left[\frac{a^{\frac{2-m}{2}}}{\frac{2-m}{2}}\right]_{a_i}^{a_c} \right)$$

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$$\therefore N - 0 = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^{m}}\left(\frac{a_c^{\frac{2-m}{2}} - a_i^{\frac{2-m}{2}}}{\frac{2-m}{2}}\right)$$

$$\therefore N = \frac{2\left(a_c^{\frac{2-m}{2}} - a_i^{\frac{2-m}{2}}\right)}{C\left(\Delta\sigma\sqrt{\pi}\right)^{m}(2-m)}$$

Where:

$$\Delta\sigma = 140MPa$$

$$a_i = 0.001m$$

$$a_c = 0.0166m$$

And typical values for C and m are:

$$m = 2.85$$

$$C = 10^{-12}\,\frac{m/cycle}{MPa\sqrt{m}}$$

Substituting these values into equation (3) gives:

$$N = 4{,}623{,}123\ cycles$$

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5. A component is made of a steel for which $K_{I_c} = 54\frac{MPa}{\sqrt{m}}$. Non-destructive testing by ultrasound methods shows that the component contains cracks up to $2a = 0.2mm$ in length. Laboratory tests show that the crack growth rate under cyclic loading is given by:

$$\frac{da}{dN} = C\Delta K^m$$

where $C = 4 \times 10^{-13}\frac{m/cycle}{MPa\sqrt{m}}$ and $m = 4$. The component is subjected to an alternating stress range, $\Delta\sigma$, of $180MPa$ about a mean tensile stress, σ_m, of $\frac{\Delta\sigma}{2}$ (i.e. $R = 0$). Given that $\Delta K = \Delta\sigma\sqrt{\pi a}$, calculate the number of cycles to failure.

[Ans: 2,404,524 cycles]

Solution 5

The applied load waveform is as shown in Fig Q5.1:

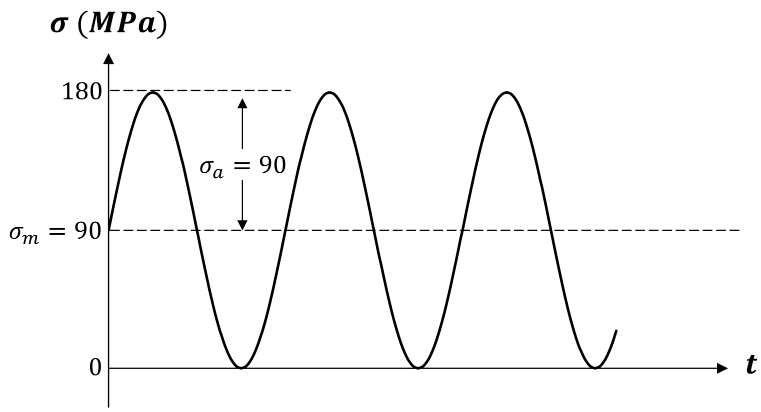

Fig 5.1 Loading waveform

$$\sigma_{nom} = 180MPa$$

It is given that:

$$K_I = \sigma_{nom}\sqrt{\pi a}$$

Therefore,

$$K_{I_c} = \sigma_{nom}\sqrt{\pi a_c}$$

(c = critical)

Where $K_{I_c} = 54\frac{MPa}{\sqrt{m}}$.

$$\therefore 54 = 180\sqrt{\pi a_c}$$

$$\therefore a_c = \left(\frac{54}{180}\right)^2 \times \frac{1}{\pi} = 0.0286m = 28.6mm$$

As given in the question:

$$\frac{da}{dN} = C\Delta K^m \tag{1}$$

and:

$$\Delta K = \Delta\sigma\sqrt{\pi a} \tag{2}$$

Substituting equation (2) into equation (1) gives:

$$da = C\left(\Delta\sigma\sqrt{\pi a}\right)^m dN$$

$$\therefore dN = \frac{1}{C\left(\Delta\sigma\sqrt{\pi a}\right)^m} da$$

$$\therefore \int_0^N dN = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\int_{a_i}^{a_c}\frac{1}{\left(\sqrt{a}\right)^m}da\left(= \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\int_{a_i}^{a_c}\frac{1}{a^{\frac{m}{2}}}da = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\int_{a_i}^{a_c}a^{-\frac{m}{2}}da\right)$$

(i = initial)

$$\therefore [N]_0^N = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\left[\frac{a^{1-\frac{m}{2}}}{1-\frac{m}{2}}\right]_{a_i}^{a_c}\left(= \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\left[\frac{a^{\frac{2-m}{2}}}{\frac{2-m}{2}}\right]_{a_i}^{a_c}\right)$$

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$$\therefore N - 0 = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\left(\frac{a_c^{\frac{2-m}{2}} - a_i^{\frac{2-m}{2}}}{\frac{2-m}{2}}\right)$$

$$\therefore N = \frac{2\left(a_c^{\frac{2-m}{2}} - a_i^{\frac{2-m}{2}}\right)}{C\left(\Delta\sigma\sqrt{\pi}\right)^m(2-m)}$$

Where:

$$\Delta\sigma = 180MPa$$

$$a_i = 0.0001m$$

$$a_c = 0.0286m$$

$$m = 4$$

$$C = 4 \times 10^{-13}\,\frac{m/cycle}{MPa\sqrt{m}}$$

Substituting these values into equation (3) gives:

$$N = 2{,}404{,}524\ cycles$$